L7 (5.4 & 5.5) Optimization Problems with Trig Functions

Example 1:

a) Find all max & mins values of the given function for $-2\pi \le \theta \le 2\pi$

b) State the intervals of INC/DEC.

c) Sketch the function

$$f(\theta) = \sin\theta + \cos\theta$$

a) $f'(\theta) = \cos\theta - \sin\theta$

Set $f'(\theta) = 0$ to find critical values

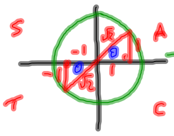
$0 = \cos\theta - \sin\theta$

$\sin\theta = \cos\theta$

$\tan\theta = 1$

Tanθ is +ve:

$\theta = \tan^{-1}(1)$



$\theta = \frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{7\pi}{4}, \frac{5\pi}{4}, \ldots$

-OR- $\theta = \frac{\pi}{4} + n\pi, n \in I$ but $-2\pi \le \theta \le 2\pi$

So... Next get y-values:

$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

$f\left(-\frac{3\pi}{4}\right) = -\sqrt{2}$

$f\left(-\frac{7\pi}{4}\right) = \sqrt{2}$

∴ Max values are $\sqrt{2}$ & Min values are $-\sqrt{2}$

b)

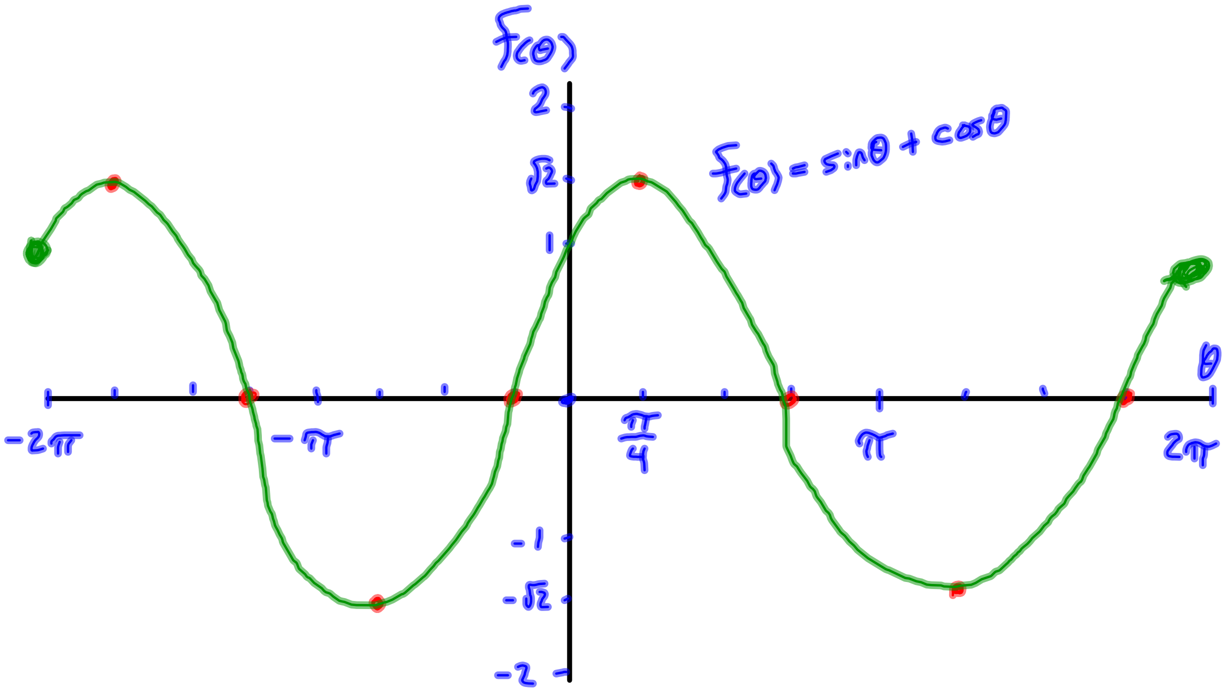
	pick -2π	pick $-\pi$	pick 0	pick $\frac{\pi}{2}$	pick $\frac{5\pi}{2}$
$f'(\theta)$	$\left(-2\pi, -\frac{7\pi}{4}\right)$	$\left(-\frac{7\pi}{4}, -\frac{3\pi}{4}\right)$	$\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$	$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	$\left(\frac{5\pi}{4}, 2\pi\right)$
$\cos\theta - \sin\theta$	+	−	+	−	+
$f(\theta)$	INC	DEC	INC	DEC	INC

MAX MIN MAX MIN

∴ The function is increasing on $\left(-2\pi, -\frac{7\pi}{4}\right)$, $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$ & $\left(\frac{5\pi}{4}, 2\pi\right)$

& decreasing on $\left(-\frac{7\pi}{4}, -\frac{3\pi}{4}\right)$ & $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

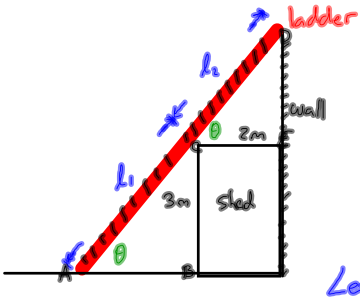
c) Sketch:


$f(\theta)$
2
$\sqrt{2}$
1
-1
$-\sqrt{2}$
-2
$f(\theta) = \sin\theta + \cos\theta$
θ
-2π
$-\pi$
$\frac{\pi}{4}$
π
2π

Example 2: A storage shed 3 meters high and 2 meters from front to back, is located beside a high vertical wall.

a) Find the length of the shortest ladder that can reach the ground, over the shed, to the wall above.

b) Find the angle this ladder makes with the ground.



Let θ represent the angle between the ground as well as the ladder and the top of the shed.

Let $\ell = \ell_1 + \ell_2$ be the entire length of the ladder.

In $\triangle ABC$

① $\sin\theta = \frac{3}{\ell_1} \Rightarrow \ell_1 = \frac{3}{\sin\theta}$

In $\triangle CDE$

② $\cos\theta = \frac{2}{\ell_2} \Rightarrow \ell_2 = \frac{2}{\cos\theta}$

Need to OPTIMIZE length of ladder ("shortest")

$$\ell(\theta) = \frac{3}{\sin\theta} + \frac{2}{\cos\theta}$$

$$= 3\csc\theta + 2\sec\theta$$

$$\ell'(\theta) = -3\csc\theta\cot\theta + 2\sec\theta\tan\theta$$

Set $\ell'(\theta) = 0$

$$0 = \frac{-3\cos\theta}{\sin^2\theta} + \frac{2\sin\theta}{\cos^2\theta}$$

$$\frac{3\cos\theta}{\sin^2\theta} = \frac{2\sin\theta}{\cos^2\theta}$$

$$3\cos^3\theta = 2\sin^3\theta$$

$$\frac{3}{2} = \frac{\sin^3\theta}{\cos^3\theta}$$

$$\tan^3\theta = \frac{3}{2}$$

$$\tan\theta = \sqrt[3]{\frac{3}{2}}$$

$$\theta = \tan^{-1}\left(\sqrt[3]{\frac{3}{2}}\right)$$

$$\theta \doteq 48.86$$

⋮

Homework: See Handout
Also Textbook p.257 #7-13 & p.260 #6,7