7.3 Product and Quotient Laws of Logarithms

The Exponent Laws:

- Product Law: $a^x a^y = a^{x+y}$
- Quotient Law: $a^x \div a^y = a^{x-y}$
- Power Law: $(a^x)^y = a^{xy}$

We have already looked at the **Power Law for Logarithms**, that is: $\log_a(m^n) = n\log_a m$

Now we are going to look at the **Product and Quotient Laws for Logarithms**:

Using the exponent Laws ⟶ $2^7 \times 2^3 = 2^{10}$ since bases are the same, you add the exponents
$7 + 3 = 10$ ①

Rewrite each (all three) of the powers in logarithmic form:

$2^7 = 128$ $\quad 2^3 = 8$ $\quad 2^{10} = 1024$

$\log_2 128 = 7$ $\quad \log_2 8 = 3$ $\quad \log_2 1024 = 10$

Then, from ① LS = $7 + 3$ RS = 10

$= \log_2 128 + \log_2 8$ $\quad = \log_2 1024$

Therefore, $(\log_2 128) + (\log_2 8) = \log_2 1024 \Rightarrow \log_2(128 \times 8)$

$= \log_2 128 + \log_2 8$

And, in general,

Product Law of Logarithms
$\log_b(mn) = \log_b m + \log_b n$

Proof of the Product Law of Logarithms:

Let $x = \log_b m$ and $y = \log_b n$ ⟹ Write these in exponential form: $m = b^x$, $n = b^y$

Then **mn** = $b^x * b^y$ (Simplify using exponent laws and rewrite in logarithmic form)

$mn = b^{x+y}$

$\log_b(mn) = x + y$

$$\boxed{\log_b(mn) = \log_b m + \log_b n}$$

as req'd

The quotient law could be developed and proven in a similar manner (left for you in question #20)

Quotient Law of Logarithms
$\log_b(m \div n) = \log_b m - \log_b n$

$y = \log x$

Example #1:
Simplify, using the laws of logarithms.
Then evaluate, correct to three decimal places.

a) $\log 99 - \log 9$

$= \log \frac{99}{9}$

$= \log 11$

$\doteq 1.041$

b) $\log_5 20 + \log_5 3$

$= \log_5 (20 \times 3)$

$= \log_5 60$

$= \frac{\log 60}{\log 5}$

$\doteq 2.544$

Example #2:
Simplify each algebraic expression.
State any restrictions on the variables.

$\log_4 x + \log_4(2y) - \log_4(z-3)$

$= \log_4 \frac{2xy}{z-3}$

~~$z \neq 3$~~

both x & $y > 0$

$z > 3$

Example #3: Evaluate, using the laws of logarithms.

a) $\log_3 54 - \log_3 6$

$= \log_3 \frac{54}{6}$

$= \log_3 9$

$= \log_3 3^2$

$= 2$

b) $\log_6 4 + \log_6 6 + \log_6 \frac{3}{2}$

$= \log_6 \left(4 \times 6 \times \frac{3}{2}\right)$

$= \log_6 36$

$= \log_6 6^2$

$= 2$

c) $\log 5 + 2\log 4 - \log 8$

$= \log 5 + \log 4^2 - \log 8$

$= \log \left(\frac{5 \times 16}{8}\right)$

$= \log 10$

$= 1$

Example #4: Simplify. State any restrictions on the variables.

a) $\log(m^5) - \log(m^2) + \log m$

$= \log \frac{m^5 m}{m^2}$

$= \log m^4$, $m > 0$

b) $\log(\sqrt[3]{p}) + \log(\sqrt{p}) + \log(\sqrt[6]{p})$

$= \log \sqrt[3]{p}(\sqrt{p})(\sqrt[6]{p})$

$= \log p^{\frac{1}{3}} p^{\frac{1}{2}} p^{\frac{1}{6}}$

$= \log p^{\frac{6}{6}} = \log p$

$p > 0$

c) $\log(6x^2 + 5x - 6) + \log(2x - 3) - \log(4x^2 - 9)$

$= \log \frac{(6x^2+5x-6)(2x-3)}{4x^2-9}$

$= \log \frac{\cancel{(2x+3)}(3x-2)\cancel{(2x-3)}}{\cancel{(2x-3)}\cancel{(2x+3)}}$

$= \log(3x-2)$, $x > \frac{3}{2}$

$2x+3>0 \Rightarrow x > -\frac{3}{2}$

$2x-3>0 \Rightarrow x > \frac{3}{2}$

$3x-2>0 \Rightarrow x > \frac{2}{3}$

Homework: pg 385; #1 – 7, 9, 10, 18, 20
Handin: pg 386 #19